

Shifted Appell sequences in Clifford analysis

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Abstract

This paper is a continuation of [D. Peña Peña, On a sequence of monogenic polynomials satisfying the Appell condition whose first term is a non-constant function, arXiv:1102.1833], in which we prove that for every monogenic polynomial $\mathbf{P}_k(x)$ of degree k in \mathbb{R}^{m+1} there exists a sequence of monogenic polynomials $\{M_n(x)\}_{n \geq 0}$ satisfying the Appell condition such that $M_0(x) = \mathbf{P}_k(x)$.

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1 Preliminaries

The real Clifford algebra $\mathbb{R}_{0,m}$ (see [6]) is the free algebra generated by the standard basis $\{e_1, \dots, e_m\}$ of the Euclidean space \mathbb{R}^m , subject to the multiplication relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, \dots, m,$$

where δ_{jk} denotes the Kronecker delta. The dimension of the real Clifford algebra $\mathbb{R}_{0,m}$ is 2^m , as is the case for the Grassmann algebra generated by $\{e_1, \dots, e_m\}$, but the difference is that now $e_j^2 = -1$ instead of $e_j^2 = 0$, creating a structure with similarities to the complex numbers.

A general element a of $\mathbb{R}_{0,m}$ may be written as $a = \sum_A a_A e_A$, $a_A \in \mathbb{R}$, in terms of the basic elements $e_A = e_{j_1} \dots e_{j_k}$, defined for every subset $A = \{j_1, \dots, j_k\}$ of $\{1, \dots, m\}$ with $j_1 < \dots < j_k$. For the empty set, one puts

$e_\emptyset = 1$, the latter being the identity element. Conjugation in $\mathbb{R}_{0,m}$ is given by $\bar{a} = \sum_A a_A \bar{e}_A$, with $\bar{e}_A = \bar{e}_{j_k} \dots \bar{e}_{j_1}$, $\bar{e}_j = -e_j$, $j = 1, \dots, m$.

One natural way to generalize the holomorphic functions to higher dimensions is by considering the null solutions of the fundamental first order differential operator ∂_x in \mathbb{R}^{m+1} given by

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}} = \partial_{x_0} + \sum_{j=1}^m e_j \partial_{x_j},$$

called the generalized Cauchy-Riemann operator, and where $\partial_{\underline{x}}$ is the Dirac operator in \mathbb{R}^m . That is, an $\mathbb{R}_{0,m}$ -valued function f defined and continuously differentiable in an open set Ω of \mathbb{R}^{m+1} , is said to be (left) monogenic in Ω if and only if $\partial_x f = 0$ in Ω . In a similar way one also defines monogenicity with respect to the Dirac operator $\partial_{\underline{x}}$ in \mathbb{R}^m . Monogenic functions are a central object of study in Clifford analysis (see e.g. [4, 8, 13]).

An remarkable feature of the generalized Cauchy-Riemann operator ∂_x is that it gives a factorization of the Laplacian, i.e.

$$\Delta_x = \sum_{j=0}^m \partial_{x_j}^2 = \partial_x \bar{\partial}_x = \bar{\partial}_x \partial_x,$$

and therefore every monogenic function is also harmonic. Observe that the operator $\bar{\partial}_x = \partial_{x_0} - \partial_{\underline{x}}$ may be seen as the higher dimensional version of the well-known operator $2\partial_z = \partial_{x_0} - i\partial_{x_1}$. Furthermore, according to [12, 15], the hypercomplex derivative of a monogenic function f is defined as

$$\frac{1}{2} \bar{\partial}_x f.$$

As a monogenic function f clearly satisfies $\partial_{x_0} f = -\partial_{\underline{x}} f$, it easily follows that

$$\frac{1}{2} \bar{\partial}_x f = \partial_{x_0} f = -\partial_{\underline{x}} f.$$

An important class of polynomial sequences is the class of Appell sequences which is defined as follows (see [1]). A polynomial sequence $\{p_n(t)\}_{n \geq 0}$, i.e. the index of each polynomial equals its degree, is said to be an Appell sequence if it satisfies

$$p'_n(t) = np_{n-1}(t), \quad n \geq 1.$$

Probably, the simplest example of an Appell sequence is the sequence $\{t^n\}_{n \geq 0}$, other examples being the Bernoulli, the Euler and the Hermite polynomials.

Appell sequences have been recently introduced to the Clifford analysis setting (see e.g. [2, 3, 5, 7, 9, 10, 11, 14, 16]). Namely, a sequence $\{P_n(x)\}_{n \geq 0}$ of $\mathbb{R}_{0,m}$ -valued polynomials forms an Appell sequence if the following conditions are satisfied:

- (i) $\{P_n(x)\}_{n \geq 0}$ is a polynomial sequence;
- (ii) each $P_n(x)$ is monogenic in \mathbb{R}^{m+1} , i.e. $\partial_x P_n(x) = 0$ for all $x \in \mathbb{R}^{m+1}$;
- (iii) $\frac{1}{2} \bar{\partial}_x P_n(x) = n P_{n-1}(x)$, $n \geq 1$.

Note that the requirement of $\{P_n(x)\}_{n \geq 0}$ being a polynomial sequence implies that the first term $P_0(x)$ must be a constant. It is natural to ask whether one can construct sequences of monogenic polynomials satisfying the Appell condition (iii) but in which the first term is a monogenic polynomial in \mathbb{R}^{m+1} and not necessarily a constant. More precisely, we are interested in sequences $\{M_n(x)\}_{n \geq 0}$ of $\mathbb{R}_{0,m}$ -valued polynomials which are monogenic in \mathbb{R}^{m+1} fulfilling

$$\frac{1}{2} \bar{\partial}_x M_n(x) = n M_{n-1}(x), \quad n \geq 1, \quad (1)$$

where $M_0(x)$ is an arbitrary monogenic polynomial in \mathbb{R}^{m+1} . These sequences will be called *shifted Appell sequences of monogenic polynomials*.

This paper is a continuation of [18], where an example of these sequences was constructed for the case $M_0(x) = \mathbf{P}_k(\underline{x})$ being an arbitrary $\mathbb{R}_{0,m}$ -valued homogeneous monogenic polynomial of degree k in \mathbb{R}^m .

It is clear that the class of shifted Appell sequences of monogenic polynomials is a right $\mathbb{R}_{0,m}$ -module under the usual addition of sequences and multiplication by Clifford numbers. Suppose now that $\mathbf{P}_\kappa(x)$ is an $\mathbb{R}_{0,m}$ -valued polynomial of degree κ which moreover is monogenic in \mathbb{R}^{m+1} . It is easy to check that $\mathbf{P}_\kappa(x)$ may be written as

$$\mathbf{P}_\kappa(x) = \sum_{k=0}^{\kappa} \mathbf{P}_k(x),$$

where $\mathbf{P}_k(x)$ denotes a homogeneous monogenic polynomial of degree k in \mathbb{R}^{m+1} . Thus, on account of the previous remarks, we only need to prove:

Theorem 1 *Let $\mathbf{P}_k(x)$ be an $\mathbb{R}_{0,m}$ -valued homogeneous polynomial of degree k which is monogenic in \mathbb{R}^{m+1} . Then there exists a shifted Appell sequence of monogenic polynomials $\{M_n(x)\}_{n \geq 0}$ such that $M_0(x) = \mathbf{P}_k(x)$.*

2 Some fundamental results

Let $\mathbf{P}(k)$ ($k \in \mathbb{N}_0$) be the set of all $\mathbb{R}_{0,m}$ -valued homogeneous polynomials of degree k in \mathbb{R}^m . This set contains the important subspace $\mathbf{M}^+(k)$ consisting of all polynomials in $\mathbf{P}(k)$ which are monogenic. That is, $\mathbf{P}_k(\underline{x}) \in \mathbf{M}^+(k)$ if it is an $\mathbb{R}_{0,m}$ -valued polynomial of degree k and

$$\mathbf{P}_k(t\underline{x}) = t^k \mathbf{P}_k(\underline{x}), \quad \partial_{\underline{x}} \mathbf{P}_k(\underline{x}) = 0, \quad \underline{x} \in \mathbb{R}^m, \quad t \in \mathbb{R}.$$

For a differentiable \mathbb{R} -valued function ϕ and a differentiable $\mathbb{R}_{0,m}$ -valued function g , we have

$$\partial_{\underline{x}}(\phi g) = \partial_{\underline{x}}(\phi)g + \phi(\partial_{\underline{x}}g). \quad (2)$$

Moreover, for a differentiable vector-valued function $\underline{f} = \sum_{j=1}^m f_j e_j$, we also have

$$\partial_{\underline{x}}(\underline{f}g) = (\partial_{\underline{x}}\underline{f})g - \underline{f}(\partial_{\underline{x}}g) - 2 \sum_{j=1}^m f_j(\partial_{x_j}g). \quad (3)$$

Let

$$\beta_k(n) = \begin{cases} n, & \text{if } n \text{ even} \\ 2k + m + n - 1, & \text{if } n \text{ odd} \end{cases}$$

for $n \geq 1$. Using the Leibniz rules (2)-(3) as well as Euler's theorem for homogeneous functions, we can deduce the useful identity:

$$\partial_{\underline{x}}(\underline{x}^n \mathbf{P}_k(\underline{x})) = -\beta_k(n) \underline{x}^{n-1} \mathbf{P}_k(\underline{x}), \quad \mathbf{P}_k(\underline{x}) \in \mathbf{M}^+(k), \quad n \geq 1. \quad (4)$$

Let us recall two basic results of Clifford analysis: the Cauchy-Kovalevskaya extension technique (see e.g. [4, 8]) and the Almansi-Fischer decomposition (see e.g. [8, 17]).

Theorem 2 *Every $\mathbb{R}_{0,m}$ -valued function $g(\underline{x})$ analytic in \mathbb{R}^m has a unique monogenic extension $\mathbf{CK}[g]$ to \mathbb{R}^{m+1} , which is given by*

$$\mathbf{CK}[g(\underline{x})](x) = \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \partial_{\underline{x}}^j g(\underline{x}).$$

Remark: Observe that a monogenic function $f(x)$ can be reconstructed by knowing its restriction to \mathbb{R}^m using previous formula, i.e.

$$f(x) = \mathbf{CK}[f(x)|_{x_0=0}](x).$$

It is also worth noting that

$$\frac{1}{2} \bar{\partial}_x \mathbf{CK}[g(\underline{x})](x) = -\partial_{\underline{x}} \mathbf{CK}[g(\underline{x})](x) = -\mathbf{CK}[\partial_{\underline{x}} g(\underline{x})](x). \quad (5)$$

Theorem 3 *Let $k \in \mathbb{N}$. Then*

$$P(k) = \bigoplus_{\nu=0}^k \underline{x}^\nu M^+(k - \nu).$$

Theorems 2 and 3 together with equality (4) will be essential for proving our main result.

3 Proof of the main result

We shall first introduce a collection $\{\{M_n^{k,\nu}(x)\}_{n \geq 0}, 0 \leq \nu \leq k\}$ of shifted Appell sequences of homogeneous monogenic polynomials whose first terms are

$$M_0^{k,\nu}(x) = \text{CK}[\underline{x}^\nu \mathbf{P}_{k-\nu}(\underline{x})](x), \quad \mathbf{P}_{k-\nu}(\underline{x}) \in M^+(k - \nu), \quad 0 \leq \nu \leq k.$$

From Theorem 2 we can deduce that $M_0^{k,\nu}(x)$ is a homogeneous monogenic polynomials of degree k in \mathbb{R}^{m+1} and is of the form

$$M_0^{k,\nu}(x) = \left(\sum_{j=0}^{\nu} \frac{\mu_j^{k,\nu}}{j!} x_0^j \underline{x}^{\nu-j} \right) \mathbf{P}_{k-\nu}(\underline{x}),$$

with $\mu_j^{k,\nu} = \prod_{s=\nu-j+1}^{\nu} \beta_{k-\nu}(s)$ for $1 \leq j \leq \nu$ and $\mu_j^{k,\nu} = 1$ for $j = 0$.

Lemma 1 *Assume that $\mathbf{P}_{k-\nu}(\underline{x}) \in M^+(k - \nu)$ where $k, \nu \in \mathbb{N}_0$, $\nu \leq k$ and put*

$$\lambda_n^{k,\nu} = \frac{n!}{\prod_{s=1}^n \beta_{k-\nu}(\nu + s)},$$

for $n \geq 1$ and $\lambda_n^{k,\nu} = 1$ for $n = 0$. The sequence $\{M_n^{k,\nu}(x)\}_{n \geq 0}$ defined by

$$M_n^{k,\nu}(x) = \lambda_n^{k,\nu} \text{CK}[\underline{x}^{\nu+n} \mathbf{P}_{k-\nu}(\underline{x})](x), \quad n \geq 0,$$

is a shifted Appell sequence of homogeneous monogenic polynomials.

Proof. By Theorem 2, it follows that $M_n^{k,\nu}(x)$ is a homogeneous monogenic polynomials of degree $k + n$ in \mathbb{R}^{m+1} and is of the form

$$M_n^{k,\nu}(x) = n! \left(\sum_{j=0}^{n-1} \frac{\lambda_{n-j}^{k,\nu}}{j!(n-j)!} x_0^j \underline{x}^{\nu+n-j} + \sum_{j=n}^{\nu+n} \frac{\mu_{j-n}^{k,\nu}}{j!} x_0^j \underline{x}^{\nu+n-j} \right) \mathbf{P}_{k-\nu}(\underline{x}), \quad n \geq 1.$$

It only remains to show that $\{\mathbf{M}_n^{k,\nu}(x)\}_{n \geq 0}$ satisfies the Appell condition (1). Indeed, using (5) and identity (4), we see at once that

$$\begin{aligned} \frac{1}{2} \bar{\partial}_x \mathbf{M}_n^{k,\nu}(x) &= -\lambda_n^{k,\nu} \text{CK}[\partial_{\underline{x}}(\underline{x}^{\nu+n} \mathbf{P}_{k-\nu}(\underline{x}))](x) \\ &= \beta_{k-\nu}(\nu+n) \lambda_n^{k,\nu} \text{CK}[\underline{x}^{\nu+n-1} \mathbf{P}_{k-\nu}(\underline{x})](x) \\ &= n \lambda_{n-1}^{k,\nu} \text{CK}[\underline{x}^{\nu+n-1} \mathbf{P}_{k-\nu}(\underline{x})](x) = n \mathbf{M}_{n-1}^{k,\nu}(x). \quad \square \end{aligned}$$

Remark: It should be noticed that $\{\mathbf{M}_n^{k,0}(x)\}_{n \geq 0}$, which is the first sequence in the above collection, corresponds to the sequence constructed in [18].

We can now prove the main result of this paper:

Proof of Theorem 1. Suppose that $\mathbf{P}_k(x)$ is a $\mathbb{R}_{0,m}$ -valued homogeneous polynomial of degree k which is monogenic in \mathbb{R}^{m+1} . From Theorem 2 we have that

$$\mathbf{P}_k(x) = \text{CK}[\mathbf{P}_k(x)|_{x_0=0}](x).$$

It is clear that $\mathbf{P}_k(x)|_{x_0=0} \in \mathbf{P}(k)$. Consequently, by Theorem 3, there exists unique $\mathbf{P}_{k-\nu}(\underline{x}) \in \mathbf{M}^+(k-\nu)$ such that

$$\mathbf{P}_k(x)|_{x_0=0} = \sum_{\nu=0}^k \underline{x}^\nu \mathbf{P}_{k-\nu}(\underline{x}).$$

From the above it follows that

$$\mathbf{P}_k(x) = \sum_{\nu=0}^k \text{CK}[\underline{x}^\nu \mathbf{P}_{k-\nu}(\underline{x})](x).$$

Define

$$\{M_n(x)\}_{n \geq 0} = \left\{ \sum_{\nu=0}^k \mathbf{M}_n^{k,\nu}(x) \right\}_{n \geq 0}.$$

Lemma 1 now shows that $\{M_n(x)\}_{n \geq 0}$ is a shifted Appell sequence of monogenic polynomials with first term $M_0(x) = \mathbf{P}_k(x)$. \square

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